

# A (2+1)–dimensional Gaussian field as fluctuations of quantum random walks on quantum groups

Jeffrey Kuan

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## Abstract

This paper introduces a (2+1)–dimensional Gaussian field which has the Gaussian free field on the upper half–plane with zero boundary conditions as certain two–dimensional sections. Along these sections, called space–like paths, it matches the Gaussian field from eigenvalues of random matrices and from a growing random surface. However, along time–like paths the behavior is different.

The Gaussian field arises as the asymptotic fluctuations in quantum random walks on quantum groups  $\mathcal{U}_q(\mathfrak{gl}_n)$ . This quantum random walk is a  $q$ –deformation of previously considered quantum random walks. The construction is accomplished utilizing Etingof–Kirillov difference operators in place of differential operators on  $GL(n)$ . When restricted to the space–like paths, the moments of the quantum random walk match the moments of the growing random surface.

## 1 Introduction

The Gaussian free field (GFF) is a two–dimensional Gaussian field which arises as asymptotics in many probabilistic models. See [22] for a mathematical introduction to the GFF. For time–dependent models it is natural to ask if there is a canonical (2+1)–dimensional Gaussian field generalizing the GFF. For a random surface growth model [7], the fluctuations along space–like paths (that is, paths in space–time where time increases as the vertical co–ordinate decreases) were shown to be the Gaussian free field – however, the behavior along time–like paths was inaccessible. In a later paper [5] which analyzed eigenvalues of corners of time–dependent random matrices, the resulting asymptotics were shown to be a time–dependent (2+1)–dimensional Gaussian field  $\mathfrak{G}$  whose restrictions to space–like paths are the GFF, and matches the [7] asymptotics along space–like paths. Additional work looking at the edge of this model was also done in [23]. These were partially motivated by physics literature, which predicted the GFF as the stationary distribution for the Anisotropic Kardar–Parisi–Zhang equation (see e.g. [16, 17, 24]).

In [19], a quantum random walk was constructed whose moments match the random surface growth model along space–like paths, and again the field  $\mathfrak{G}$  arises in the asymptotics, after applying the standard Brownian Motion to Ornstein–Uhlenbeck rescaling. It is natural to ask if  $\mathfrak{G}$  is the only canonical time–dependent Gaussian field having the GFF as fixed–time marginals (a very rough analogy would be the characterization of the Ornstein–Uhlenbeck process as the only Gaussian, stationary, Markov process). However, this paper will construct a different field  $\mathcal{G}$ .

The field  $\mathcal{G}$  will again arise as fluctuations of a quantum random walk on quantum groups (QRWQG), which is a variant of [1, 3, 6, 9, 19, 20]. As was the case in [19], along space–like

paths the moments of the QRWQG are precisely the same as the moments for the random surface growth model from [7]. This shows that  $\mathfrak{G}$  and  $\mathcal{G}$  are identical along space-like paths, but it turns out that they are not the same along time-like paths.

Having introduced  $\mathcal{G}$ , now turn the discussion to the quantum random walks. Recall that the motivation for quantum random walks comes from quantum mechanics. Rather than defining a state space as a set of states, instead the state space is a Hilbert space of wavefunctions. The observables, rather than being functions on the state space, are operators on this Hilbert space. These operators are related to classical observables through their eigenvalues. Generally, observables do not have to commute, so for this reason quantum random walks are also called non-commutative random walks. The randomness occurs through *states*, which are linear functionals on the space of observables, corresponding to the expectation of the observable.

Before describing the quantum version of the random surface, first review how it looks in the classical viewpoint. On each horizontal section of the random stepped surface, the (classical) state space is the set

$$\{\lambda = (\lambda_1 \geq \dots \geq \lambda_n) : \lambda_i \in \mathbb{Z}\}$$

By analogy with quantum mechanics, the space of wavefunctions should consist of functions of the form  $\chi_{\vec{x}}(\lambda)$ . By dimension considerations,  $\vec{x}$  should vary over  $\mathbb{C}^n$ . Switching indices and variables, write  $\chi_{\lambda}(x_1, \dots, x_n)$ , so the wave functions are some class of functions on  $\mathbb{C}^n$ . The observables are then some space of operators on this class of functions. Any probability measure  $\mathbb{P}(\lambda)$  on the  $\lambda$  can be encoded through  $\chi = \sum_{\lambda} \mathbb{P}(\lambda) \chi_{\lambda}$ . Furthermore, if  $D$  is an operator for which  $\chi_{\lambda}$  are eigenfunctions with eigenvalues  $a(\lambda)$ , then

$$(D\chi)(1, \dots, 1) = \sum_{\lambda} \mathbb{P}(\lambda) a(\lambda) \chi_{\lambda}(1, \dots, 1) = \mathbb{E}_{\mathbb{P}}[a(\lambda) \cdot \chi_{\lambda}(1, \dots, 1)]$$

The  $\chi_{\lambda}$  can be chosen so that  $\chi_{\lambda}(1, \dots, 1)$  is normalized to 1. The phenomenon that a state can be defined from a wave function can be seen as an analog of the Gelfand–Naimark–Segal construction.

The actual construction of the observables and  $\chi_{\lambda}$  comes from representation theory. In particular, the set of such  $\lambda$  parameterizes the highest weights of finite-dimensional irreducible representations of  $\mathfrak{gl}_n$ , and it is through these representations that the observables are defined. Here, the (non-commutative) space of observables is the Drinfeld–Jimbo quantum group  $\mathcal{U}_q(\mathfrak{gl}_n)$ . Each  $u \in \mathcal{U}_q(\mathfrak{gl}_n)$  has a corresponding difference operator from [11]. The relevant observables for asymptotics are certain central elements  $C_q^{(n)} \in Z(\mathcal{U}_q(\mathfrak{gl}_n))$  calculated in [13]. The eigenvalue of each  $C_q^{(n)}$  on the irreducible representation  $V_{\lambda}$  is  $\sum_{i=1}^n q^{2(\lambda_i - i + n)}$ , so one can think of these observables as linear statistics of the function  $q^{2x}$ .

There are a few key differences between this construction and previous constructions that are worth highlighting. When  $q \rightarrow 1$ , the QRWQG reduces to the one in [6, 19], which used the universal enveloping algebra  $\mathcal{U}(\mathfrak{gl}_n)$  as the space of observables. The papers there used differential operators on the Lie group  $GL(n)$  to define the quantum random walk. The relevant central elements in  $\mathcal{U}(\mathfrak{gl}_n)$  acted as  $\sum_{i=1}^n (\lambda_i - i + n)^k$  for  $k \geq 1$ , so can be thought of as linear statistics of  $x^k$ . In that case, the fluctuations of linear statistics for different values of  $k$  were computed, which suggests finding the fluctuations of these linear statistics for different  $q$  here.

Additionally,  $\mathcal{U}_q(\mathfrak{gl}_n)$  is no longer co-commutative as a Hopf algebra. (The word “quantum” appearing twice in the title of this paper refers to two different meanings: the first one makes the space of observables non-commutative, and the second makes it non-co-commutative). Probabilistically, this results in different dynamics, which ultimately leads to the Gaussian

fields differing along time-like paths. However, the functions  $\chi_\lambda$  do not depend on  $q$ , which is why the Gaussian fields match along space-like paths.

We also mention several algebraic reasons for taking the approach with quantum groups. One is that in the  $q \rightarrow 1$  limit, the asymptotics are dependent on Schur-Weyl duality (see equation (2.9) of [6], which references Proposition 3.7 of [14]), so does not generalize to other Lie algebras. Additionally, the relevant central elements are actually easier to construct in the quantum case than in the classical case (for example, see section 7.5 of [12] or chapter 7 of [21] for explicit central elements of  $\mathcal{U}(\mathfrak{gl}_n)$ ). Another notable difference is that the non-commutative Markov operator  $P_t$  no longer preserves the center when  $q \neq 1$ . However, (somewhat surprisingly) each  $P_t C_q^{(n)}$  can be written as a linear combination of  $C_q^{(n-k)}$ , generalizing a result of [4] for  $n = 2$ . So this paper demonstrates (in a sense) that preserving the center is not necessary for developing meaningful asymptotics. However, note that one would not necessarily expect  $P_t C_q^{(n)}$  to be a linear combination of  $C_q^{(n-k)}$  in every quantum group, so while it should be possible to construct a QRWQG in general, the asymptotics may be more difficult.

Finally, it is important to mention that some of the cited papers actually prove more than what is needed here. For example, [5] and [6] actually show convergence to correlated Gaussian free fields. The difference operators in [11] can be used to construct Macdonald's difference operators for all  $(q, t)$ , and here only the  $q = t$  case is used. Therefore, it should be possible to extend the results of this paper to more generality.

Let us outline the body of the paper. In section 2, we define the Gaussian field  $\mathcal{G}$  and show by direct computation its relationship to  $\mathfrak{G}$  along space-like paths. Section 3 reviews some of the necessary definitions in non-commutative probability theory and representation theory. Section 4 provides the construction of the quantum random walk. In section 5, the random surface growth from [7] is defined and shown to have the same expectations as the quantum random walk along space-like paths. Finally, section 6 shows that the fluctuations in the QRWQG converge to  $\mathcal{G}$ .

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## 2 A (2+1)–dimensional Gaussian field

Let  $\mathfrak{G}$  be a Gaussian field indexed by  $(k, \eta, \tau) \in \mathbb{N} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$  with mean zero and covariance given by

$$\mathbb{E}[\mathfrak{G}(k_i, \eta_i, \tau_i) \mathfrak{G}(k_j, \eta_j, \tau_j)] = \begin{cases} \left(\frac{1}{2\pi i}\right)^2 \int \int_{|z| > |w|} (\eta_i z^{-1} + \tau_i + \tau_i z)^{k_i} (\eta_j w^{-1} + \tau_j + \tau_j w)^{k_j} (z - w)^{-2} dz dw, & \eta_i \geq \eta_j, \tau_i \leq \tau_j \\ \left(\frac{1}{2\pi i}\right)^2 \int \int_{|z| > |w|} (\eta_j \frac{\tau_j}{\tau_i} z^{-1} + \tau_j + \tau_i z)^{k_j} (\eta_i w^{-1} + \tau_i + \tau_i w)^{k_i} (z - w)^{-2} dz dw, & \eta_i < \eta_j, \tau_i \leq \tau_j \end{cases}$$

where the  $z, w$  contours are counterclockwise circles centered around the origin. As explained in [19],  $\mathfrak{G}$  can be viewed as the moments of a three-dimensional Gaussian field which has the

Gaussian free field as fixed-time marginals.

Let  $\mathcal{G}$  be a Gaussian field indexed by  $(h, \eta, \tau) \in \mathbb{C} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$  with mean zero and covariance  $\mathbb{E}[\mathcal{G}(\tilde{h}, \eta_i, \tau_i) \mathcal{G}(h, \eta_j, \tau_j)]$  given by:

$$\begin{aligned} & \text{if } \tau_j \geq \tau_i, \eta_j \leq \eta_i, \quad \left(\frac{1}{2\pi i}\right)^2 e^{2\tilde{h}\tau_i} e^{2h\tau_j} \iint_{|z|>|w|} \exp\left(2\tilde{h}(\eta_i z^{-1} + \tau_i z)\right) (z-w)^{-2} dz dw \\ & \times \left( \exp(2h(\eta_j w^{-1} + \tau_i w)) + \frac{2h\sqrt{\tau_j - \tau_i}}{2\pi i} \int_0^{\eta_j} e^{2h((\eta_j - \kappa)w^{-1} + \tau_i w)} d\kappa \oint e^{2h\sqrt{\tau_j - \tau_i}\sqrt{\kappa}(t+t^{-1})} \frac{t^{-2}}{\sqrt{\kappa}} dt \right), \end{aligned} \quad (1)$$

$$\begin{aligned} & \text{if } \tau_j \geq \tau_i, \eta_j \geq \eta_i, \quad \left(\frac{1}{2\pi i}\right)^2 e^{2\tilde{h}\tau_i} e^{2h\tau_j} \iint_{|z|>|w|} \exp\left(2\tilde{h}(\eta_i z^{-1} + \tau_i z)\right) (z-w)^{-2} dz dw \\ & \times \left( \exp(2h(\eta_j w^{-1} + \tau_i w)) + \frac{2h\sqrt{\tau_j - \tau_i}}{2\pi i} \int_{\eta_j - \eta_i}^{\eta_j} e^{2h((\eta_j - \kappa)w^{-1} + \tau_i w)} d\kappa \oint e^{2h\sqrt{\tau_j - \tau_i}\sqrt{\kappa}(t+t^{-1})} \frac{t^{-2}}{\sqrt{\kappa}} dt \right) \\ & + \left(\frac{1}{2\pi i}\right)^2 e^{2\tilde{h}\tau_i} e^{2h\tau_j} \iint_{|z|<|w|} \exp\left(2\tilde{h}(\eta_i z^{-1} + \tau_i z)\right) (z-w)^{-2} dz dw \\ & \times \left( \frac{2h\sqrt{\tau_j - \tau_i}}{2\pi i} \int_0^{\eta_j - \eta_i} e^{2h((\eta_j - \kappa)w^{-1} + \tau_i w)} d\kappa \oint e^{2h\sqrt{\tau_j - \tau_i}\sqrt{\kappa}(t+t^{-1})} \frac{t^{-2}}{\sqrt{\kappa}} dt \right), \end{aligned} \quad (2)$$

where the  $t, z, w$  contours are counterclockwise circles centered around the origin, with  $|t| > |w|\sqrt{\tau_j - \tau_i}$ . Note that if  $\tau_i \leq \tau_j$  then

$$\begin{aligned} \mathbb{E}[\mathcal{G}(\tilde{h}, \eta_i, \tau_i) \mathcal{G}(h, \eta_j, \tau_j)] &= e^{2h(\tau_j - \tau_i)} \mathbb{E}[\mathcal{G}(\tilde{h}, \eta_i, \tau_i) \mathcal{G}(h, \eta_j, \tau_i)] \\ &+ e^{2h(\tau_j - \tau_i)} \frac{2h\sqrt{\tau_j - \tau_i}}{2\pi i} \int_0^{\eta_j} \mathbb{E}[\mathcal{G}(\tilde{h}, \eta_i, \tau_i) \mathcal{G}(h, \eta_j - \kappa, \tau_i)] d\kappa \oint e^{2h\sqrt{\tau_j - \tau_i}\sqrt{\kappa}(t+t^{-1})} \frac{t^{-2}}{\sqrt{\kappa}} dt \end{aligned} \quad (3)$$

A priori, it is not obvious that  $\mathcal{G}$  is a well-defined family of random variables: for instance, the covariance matrix might not be positive-definite. However, it will be shown in Theorem 6.3 below that  $\mathcal{G}$  occurs as the limit of well-defined random variables. Furthermore, numerical computations indicate that the covariance matrices are positive-definite anyway.

The next proposition shows that  $\mathcal{G}$  and  $\mathfrak{G}$  match along space-like paths. It also follows from later results (namely, that  $\mathcal{G}$  is the limit of the QRWQG, the QRWQG matches the surface growth along space-like paths, and  $\mathfrak{G}$  is the limit of the surface growth), but a more elementary proof is provided here. Because  $\mathcal{G}$  will appear in the linear statistics of  $q^{2x}$  and  $\mathfrak{G}$  appears as linear statistics of  $x^k$ , setting  $q = e^h$  motivates the comparison.

**Proposition 2.1.** *If  $\eta_i \geq \eta_j$  and  $\tau_i \leq \tau_j$  then*

$$\mathbb{E}[\mathcal{G}(\tilde{h}, \eta_i, \tau_i) \mathcal{G}(h, \eta_j, \tau_j)] = \sum_{k_i, k_j=0}^{\infty} \frac{(2\tilde{h})^{k_i} (2h)^{k_j}}{k_i! k_j!} \mathbb{E}[\mathfrak{G}(k_i, \eta_i, \tau_i) \mathfrak{G}(k_j, \eta_j, \tau_j)] \quad (4)$$

*If  $\tau_j \geq \tau_i, \eta_j \geq \eta_i$ , then in general (4) does not hold.*

*Proof.* Assume that  $\eta_i \geq \eta_j$  and  $\tau_i \leq \tau_j$ . By making the substitution  $t \mapsto t/\sqrt{\kappa}$ , the expression

becomes

$$\begin{aligned} & \left( \frac{1}{2\pi i} \right)^2 e^{2\tilde{h}\tau_i} e^{2h\tau_j} \iint_{|z|>|w|} \exp \left( 2\tilde{h}(\eta_i z^{-1} + \tau_i z) \right) (z-w)^{-2} dz dw \\ & \times \left( \exp(2h(\eta_j w^{-1} + \tau_i w)) + \frac{2h\sqrt{\tau_j - \tau_i}}{2\pi i} \int_0^{\eta_j} e^{2h((\eta_j - \kappa)w^{-1} + \tau_i w)} d\kappa \oint e^{2h\sqrt{\tau_j - \tau_i}(t + \kappa t^{-1})} t^{-2} dt \right) \end{aligned} \quad (5)$$

The integrand in  $\kappa$  is simply an exponential function, so evaluates to

$$\begin{aligned} & \left( \frac{1}{2\pi i} \right)^2 e^{2\tilde{h}\tau_i} e^{2h\tau_j} \iint_{|z|>|w|} \exp \left( 2\tilde{h}(\eta_i z^{-1} + \tau_i z) \right) (z-w)^{-2} dz dw \\ & \times \left( \exp(2h(\eta_j w^{-1} + \tau_i w)) + \frac{2h\sqrt{\tau_j - \tau_i}}{2\pi i} \oint \left( \frac{e^{2h\eta_j(\sqrt{\tau_j - \tau_i}t^{-1} - w^{-1})} - 1}{2h(\sqrt{\tau_j - \tau_i}t^{-1} - w^{-1})} \right) e^{2h(\eta_j w^{-1} + \tau_i w)} e^{2ht\sqrt{\tau_j - \tau_i}} t^{-2} dt \right) \end{aligned} \quad (6)$$

Substitute  $t \mapsto \sqrt{\tau_j - \tau_i}t$

$$\begin{aligned} & \left( \frac{1}{2\pi i} \right)^2 e^{2\tilde{h}\tau_i} e^{2h\tau_j} \iint_{|z|>|w|} \exp \left( 2\tilde{h}(\eta_i z^{-1} + \tau_i z) \right) (z-w)^{-2} dz dw \\ & \times \left( \exp(2h(\eta_j w^{-1} + \tau_i w)) + \frac{2h}{2\pi i} \oint \left( \frac{e^{2h\eta_j(t^{-1} - w^{-1})} - 1}{2ht(1 - tw^{-1})} \right) e^{2h(\eta_j w^{-1} + \tau_i w)} e^{2ht(\tau_j - \tau_i)} dt \right) \end{aligned} \quad (7)$$

By the assumptions on the  $t$  and  $w$  contours,

$$\frac{2h}{2\pi i} \oint \left( \frac{-1}{2ht(1 - tw^{-1})} \right) e^{2ht(\tau_j - \tau_i)} dt = -1 + e^{2hw(\tau_j - \tau_i)}$$

leaving us with

$$\begin{aligned} & \left( \frac{1}{2\pi i} \right)^2 e^{2\tilde{h}\tau_i} e^{2h\tau_j} \iint_{|z|>|w|} \exp \left( 2\tilde{h}(\eta_i z^{-1} + \tau_i z) \right) (z-w)^{-2} dz dw \\ & \times \left( \exp(2h(\eta_j w^{-1} + \tau_j w)) + \frac{1}{2\pi i} \oint \left( \frac{e^{2h\eta_j(t^{-1} - w^{-1})}}{t(1 - tw^{-1})} \right) e^{2h(\eta_j w^{-1} + \tau_i w)} e^{2ht(\tau_j - \tau_i)} dt \right) \end{aligned} \quad (8)$$

So it remains to check that

$$\begin{aligned} & \left( \frac{1}{2\pi i} \right)^2 e^{2\tilde{h}\tau_i} e^{2h\tau_j} \iint_{|z|>|w|} \exp \left( 2\tilde{h}(\eta_i z^{-1} + \tau_i z) \right) (z-w)^{-2} dz dw \\ & \times \left( \frac{1}{2\pi i} \oint \left( \frac{e^{2h\eta_j(t^{-1} - w^{-1})}}{t(1 - tw^{-1})} \right) e^{2h(\eta_j w^{-1} + \tau_i w)} e^{2ht(\tau_j - \tau_i)} dt \right) = 0 \end{aligned} \quad (9)$$

But this follows immediately, because the  $w^{-1}$  terms in the exponential cancel, so the integrand has no residues in  $w$ . So (4) is true.

Now suppose that  $\tau_j \geq \tau_i, \eta_j \geq \eta_i$ . The  $\tilde{h}h^3$  coefficient of the right-hand-side of (4) is

$$\frac{2\tilde{h}(2h)^3}{6} \cdot \eta_i \cdot (3\tau_i\tau_j^2 + 3\eta_j\tau_j\tau_i) = 8\tilde{h}h^3\eta_i\tau_i\tau_j(\tau_j + \eta_j)$$

But on the left-hand-side it is

$$\begin{aligned}
& \frac{16}{3!} \eta_j \cdot (3\tau_i \cdot \tau_i^2 + 3\eta_i \tau_i \cdot \tau_i) + 2\sqrt{\tau_j - \tau_i} \int_0^{\eta_j} 4\tau_i \min(\eta_j - \kappa, \eta_i) d\kappa \cdot 2\sqrt{\tau_j - \tau_i} \\
&= \frac{16}{6} \eta_j \cdot (3\tau_i \cdot \tau_i^2 + 3\eta_i \tau_i \cdot \tau_i) + 16(\tau_j - \tau_i) \tau_i \left( -\frac{1}{2}(\eta_j - \kappa)^2 \Big|_{\kappa=\eta_j-\eta_i}^{\eta_j} + \eta_i(\eta_j - \eta_i) \right) \\
&= 8\eta_j \cdot (\tau_i \cdot \tau_i^2 + \eta_i \tau_i \cdot \tau_i) + 16(\tau_j - \tau_i) \tau_i \left( \frac{1}{2}\eta_i^2 + \eta_i(\eta_j - \eta_i) \right)
\end{aligned}$$

which is not equal to the expression above.  $\square$

## 3 Background Definitions

### 3.1 Non-commutative probability

Here are some of the basic definitions of objects in non-commutative probability. A more comprehensive introduction can be found in [2].

A non-commutative probability space  $(\mathcal{A}, \phi)$  is a unital  $*$ -algebra  $\mathcal{A}$  with identity 1 and a state  $\phi : \mathcal{A} \rightarrow \mathbb{C}$ , that is, a linear map such that  $\phi(a^*a) \geq 0$  and  $\phi(1) = 1$ . Elements of  $\mathcal{A}$  are called *non-commutative random variables*. This generalizes a classical probability space, by considering  $\mathcal{A} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$  with  $\phi(X) = \mathbb{E}_{\mathbb{P}} X$ . We also need a notion of convergence. For a large parameter  $L$  and  $a_1, \dots, a_r \in \mathcal{A}$ ,  $\phi$  which depend on  $L$ , as well as a limiting space  $(\mathbb{A}, \Phi)$ , we say that  $(a_1, \dots, a_r)$  converges to  $(\mathbf{a}_1, \dots, \mathbf{a}_r)$  with respect to the state  $\phi$  if

$$\phi(a_{i_1}^{\epsilon_1} \dots a_{i_k}^{\epsilon_k}) \rightarrow \Phi(\mathbf{a}_{i_1}^{\epsilon_1} \dots \mathbf{a}_{i_k}^{\epsilon_k})$$

for any  $i_1, \dots, i_k \in \{1, \dots, r\}$ ,  $\epsilon_j \in \{1, *\}$  and  $k \geq 1$ .

There is also a non-commutative version of a Markov chain. If  $X_n : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow E$  denotes the Markov process with transition operator  $Q : L^\infty(E) \rightarrow L^\infty(E)$ , then the Markov property is

$$\mathbb{E}[Y f(X_{n+1})] = \mathbb{E}[Y Qf(X_n)]$$

for  $f \in L^\infty(E)$  and  $Y$  a  $\sigma(X_1, \dots, X_n)$ -measurable random variable. Letting  $j_n : L^\infty(E) \rightarrow L^\infty(\Omega, \mathcal{F}, \mathbb{P})$  be defined by  $j_n(f) = f(X_n)$ , we can write the Markov property as

$$\mathbb{E}[j_{n+1}(f)Y] = \mathbb{E}[j_n(Qf)Y]$$

for all  $f \in L^\infty(E)$  and  $Y$  in the subalgebra of  $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$  generated by the images of  $j_0, \dots, j_n$ .

Translating into the non-commutative setting, we define a *non-commutative Markov operator* to be a semigroup of unital linear maps  $\{P_t : t \in T\}$  from a  $*$ -algebra  $U$  to itself (not necessarily an algebra morphism). The set  $T$  indexing time can be either  $\mathbb{N}$  or  $\mathbb{R}_{\geq 0}$ , that is to say, the Markov process can be either discrete or continuous time. We also require that for any times  $t_0 < t_1 < \dots \in T$  there exists algebra morphisms  $j_{t_n}$  from  $U$  to a non-commutative probability space  $(W, \omega)$  such that

$$\omega(j_{t_n}(f)w) = \omega(j_{t_{n-1}}(P_{t_n-t_{n-1}}f)w)$$

for all  $f \in U$  and  $w$  in the subalgebra of  $W$  generated by the images of  $\{j_t : t \leq t_{n-1}\}$ .

### 3.2 Representation theory

#### 3.2.1 Definition of Quantum Groups

This sub-subsection defines the quantum groups. See [15] for a more thorough treatment.

Let  $q$  be a formal variable. The quantum group  $\mathcal{U}_q(\mathfrak{gl}_n)$  is the Hopf algebra with generators  $\{E_{i,i+1}, E_{i+1,i} : 1 \leq i \leq n-1\}, \{q^{E_{ii}} : 1 \leq i \leq n\}$  satisfying the relations (below,  $q^{E_{ii}}$  are all invertible and the multiplication is written additively in the exponential, so for example  $q^{-E_{11}+E_{22}} = (q^{E_{11}})^{-1} (q^{E_{22}})^2$ )

$$\begin{aligned} q^{E_{ii}} q^{E_{jj}} &= q^{E_{jj}} q^{E_{ii}} = q^{E_{ii}+E_{jj}} \\ [E_{i,i+1}, E_{i+1,i}] &= \frac{q^{E_{ii}-E_{i+1,i+1}} - q^{E_{i+1,i+1}-E_{ii}}}{q - q^{-1}} & [E_{i,i+1}, E_{j+1,j}] &= 0, \quad i \neq j \\ q^{E_{ii}} E_{i,i+1} &= q E_{i,i+1} q^{E_{ii}} & q^{E_{ii}} E_{i-1,i} &= q^{-1} E_{i-1,i} q^{E_{ii}} & [q^{E_{ii}}, E_{j,j+1}] &= 0, \quad j \neq i, i-1 \\ q^{E_{ii}} E_{i,i-1} &= q E_{i,i-1} q^{E_{ii}} & q^{E_{ii}} E_{i+1,i} &= q^{-1} E_{i+1,i} q^{E_{ii}} & [q^{E_{ii}}, E_{j,j-1}] &= 0, \quad j \neq i, i+1 \end{aligned}$$

$$\begin{aligned} E_{i,i+1}^2 E_{j,j+1} - (q + q^{-1}) E_{i,i+1} E_{j,j+1} E_{i,i+1} + E_{j,j+1} E_{i,i+1}^2 &= 0, \quad i = j \pm 1 \\ E_{i,i-1}^2 E_{j,j-1} - (q + q^{-1}) E_{i,i-1} E_{j,j-1} E_{i,i-1} + E_{j,j-1} E_{i,i-1}^2 &= 0, \quad i = j \pm 1 \\ [E_{i,i+1}, E_{j,j+1}] &= 0 = [E_{i,i-1}, E_{j,j-1}], \quad i \neq j \pm 1 \end{aligned}$$

The co-product is an algebra morphism  $\Delta : \mathcal{U}_q(\mathfrak{gl}_n) \rightarrow \mathcal{U}_q(\mathfrak{gl}_n) \otimes \mathcal{U}_q(\mathfrak{gl}_n)$  defined by

$$\begin{aligned} \Delta(q^{E_{ii}}) &= q^{E_{ii}} \otimes q^{E_{ii}} \\ \Delta(E_{i,i+1}) &= q^{E_{ii}-E_{i+1,i+1}} \otimes E_{i,i+1} + E_{i,i+1} \otimes 1 \\ \Delta(E_{i,i-1}) &= 1 \otimes E_{i,i-1} + E_{i,i-1} \otimes q^{E_{i+1,i+1}-E_{ii}} \end{aligned}$$

Note that unless  $q \rightarrow 1$ ,  $\Delta$  does not satisfy co-commutativity. In other words, if  $P$  is the permutation  $P(a \otimes b) = b \otimes a$ , then  $P \circ \Delta \neq \Delta$ . However, the co-product does satisfy the co-associativity property

$$(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta,$$

so that there is a well-defined algebra morphism  $\Delta^{(m-1)} : \mathcal{U}_q(\mathfrak{gl}_n) \rightarrow \mathcal{U}_q(\mathfrak{gl}_n)^{\otimes m}$  satisfying  $\Delta^{(m)} = (\text{id} \otimes \Delta^{(m-1)}) \circ \Delta = (\Delta^{(m-1)} \otimes \text{id}) \circ \Delta$ . Recall Sweedler's notation  $\Delta(u) = \sum u_{(1)} \otimes u_{(2)}$  and sumless Sweedler's notation  $\Delta(u) = u_{(1)} \otimes u_{(2)}$ . This notation will extend to

$$\begin{aligned} \Delta^{(n-1)}(u) &= u_{(1)} \otimes \cdots \otimes u_{(n)} \\ (\text{id} \otimes \Delta) \circ \Delta(u) &= u_{(1)} \otimes u_{(21)} \otimes u_{(22)} & (\Delta \otimes \text{id}) \circ \Delta(u) &= u_{(11)} \otimes u_{(12)} \otimes u_{(2)} \end{aligned}$$

For completeness, the antipode  $S$  is an anti-automorphism on  $\mathcal{U}_q(\mathfrak{gl}_n)$  defined on generators by

$$S(E_{i,i+1}) = -q^{-1} E_{i,i+1} \quad S(E_{i,i-1}) = -q E_{i,i-1}, \quad S(q^{E_{ii}}) = q^{-E_{ii}}.$$

and the co-unit is an algebra morphism  $\epsilon : \mathcal{U}_q(\mathfrak{gl}_n) \rightarrow \mathbb{C}$  defined on generators by

$$\epsilon(q^{E_{ii}}) = 1 \quad \epsilon(E_{i,i \pm 1}) = 0.$$

The antipode will only be used in the remark at the end of section 4, and the co-unit will not be used explicitly.

For any  $1 \leq i \neq j \leq n$ , define  $E_{ij}$  inductively by

$$E_{ij} = E_{ik} E_{kj} - q^{-1} E_{kj} E_{ik}, \quad i \leq k \leq j$$

From the relations defining  $\mathcal{U}_q(\mathfrak{gl}_n)$ , it is not hard to see that for  $1 \leq i < j \leq n$ ,

$$\begin{aligned} \Delta E_{ij} &= E_{ij} \otimes 1 + q^{E_{ii}-E_{jj}} \otimes E_{ij} + (q - q^{-1}) \left[ \sum_{r=i+1}^{j-1} (q^{E_{rr}-E_{jj}} E_{ir}) \otimes E_{rj} \right] \\ \Delta E_{ji} &= 1 \otimes E_{ji} + E_{ji} \otimes q^{E_{jj}-E_{ii}} + (q - q^{-1}) \left[ \sum_{r=i+1}^{j-1} E_{ri} \otimes (q^{E_{rr}-E_{ii}} E_{jr}) \right]. \end{aligned} \tag{10}$$

### 3.2.2 Representations

Assume that  $q$  is not a root of unity. The finite-dimensional irreducible representations of  $\mathcal{U}_q(\mathfrak{gl}_n)$  are parameterized by non-increasing sequences of  $n$  integers

$$\mathbb{GT}_n := \{(\lambda_1 \geq \dots \geq \lambda_n) : \lambda_i \in \mathbb{Z}\}$$

For each  $\lambda \in \mathbb{GT}_n$ , let  $V_\lambda$  denote the corresponding representation. There is a *weight space decomposition*

$$V_\lambda = \bigoplus_{\mu} V_\lambda[\mu]$$

where  $\mu$  is some sequence of integers  $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$  (not necessarily non-increasing) and

$$V_\lambda[\mu] = \{v \in V_\lambda : q^{a_1 E_{11} + \dots + a_n E_{nn}} v = q^{a_1 \mu_1 + \dots + a_n \mu_n} v\}$$

One can think of the weight spaces as a generalization of eigenspaces. Given any complex numbers  $x_1, \dots, x_n$  there is an action on  $V_\lambda$  by

$$x_1^{E_{11}} \dots x_n^{E_{nn}} v = x_1^{\mu_1} \dots x_n^{\mu_n} v \quad \text{for } v \in V_\lambda[\mu].$$

and write  $x^E v$  for the left-hand-side. With this notation, define the *character*  $\chi_\lambda$  as

$$\chi_\lambda(x_1, \dots, x_n) = \text{Tr}|_{V_\lambda} (x^E). \quad (11)$$

Let  $\dim \lambda = \chi_\lambda(1, \dots, 1)$  denote the dimension of  $V_\lambda$ . Each  $\chi_\lambda$  is a symmetric polynomial and the  $\{\chi_\lambda\}$  form a basis for the ring of symmetric polynomials in  $n$  variables. (in fact, these are the Schur polynomials, although this information will not be used explicitly in this paper).

The co-product defines the action on tensor products of representations, in the sense that if  $v, w$  are vectors in two different representations, then

$$u \cdot (v \otimes w) = u_{(1)} v \otimes u_{(2)} w.$$

There are also branching rules between representations of  $\mathcal{U}_q(\mathfrak{gl}_n)$  and  $\mathcal{U}_q(\mathfrak{gl}_{n-1})$ . For  $\lambda \in \mathbb{GT}_n$  and  $\mu \in \mathbb{GT}_{n-1}$ , let  $\mu \prec \lambda$  mean

$$\mu \prec \lambda \text{ if and only if } \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n.$$

If  $V_\lambda$  is restricted to  $\mathbb{GT}_{n-1}$  then it decomposes as

$$V_\lambda = \bigoplus_{\mu \prec \lambda} V_\mu$$

On the level of characters, this means that

$$\chi_\lambda(x_1, \dots, x_{n-1}, 1) = \sum_{\mu \prec \lambda} \chi_\mu(x_1, \dots, x_{n-1}).$$

More generally, if  $m \leq n$ ,  $\lambda^{(m)} \in \mathbb{GT}_m$  and  $\lambda^{(n)} \in \mathbb{GT}_n$ , then let  $m(\lambda^{(n)}, \lambda^{(m)})$  denote the multiplicities of  $V_{\lambda^{(n)}}$  restricted to  $\mathbb{GT}_m$ , that is

$$V_{\lambda^{(n)}} = \bigoplus_{\lambda^{(m)} \in \mathbb{GT}_m} m(\lambda^{(n)}, \lambda^{(m)}) V_{\lambda^{(m)}}$$

which also means that

$$\chi_{\lambda^{(n)}}(x_1, \dots, x_m, 1, \dots, 1) = \sum_{\lambda^{(m)} \in \mathbb{GT}_m} m(\lambda^{(n)}, \lambda^{(m)}) \chi_{\lambda^{(m)}}(x_1, \dots, x_m)$$

By setting  $x_1 = \dots = x_n = 1$ , this shows that

$$\sum_{\lambda^{(m)} \in \mathbb{GT}_m} \Lambda(\lambda^{(n)}, \lambda^{(m)}) = 1, \quad \text{where } \Lambda(\lambda^{(n)}, \lambda^{(m)}) = m(\lambda^{(n)}, \lambda^{(m)}) \frac{\dim \lambda^{(m)}}{\dim \lambda^{(n)}} \quad (12)$$



### 3.2.3 Etingof–Kirillov Difference Operators

Let  $M$  be the algebra of linear operators on the space of formal power series in infinitely many variables  $\mathbb{C}[[x_1, x_2, \dots]]$ . The multiplication in  $M$  is the usual composition of operators, denoted  $\circ$ . In [11], there is a definition of difference operators  $D_u \in M$  for each  $u \in \mathcal{U}_q(\mathfrak{gl}_n)$ . For the purposes of this paper, only the following properties of  $D_u$  will be used:

- If  $u \in \mathcal{U}_q(\mathfrak{gl}_n)$  then  $D_u$  only acts on functions of  $n$  variables  $f(x_1, \dots, x_n)$ , and fixes the remaining variables.
- The inclusion  $\mathcal{U}_q(\mathfrak{gl}_n) \rightarrow \mathcal{U}(\mathfrak{gl}_{n+1})$  is consistent with  $u \mapsto D_u$ . In other words,  $D_u$  is the same operator whether  $u$  is considered an element of  $\mathcal{U}_q(\mathfrak{gl}_n)$  or  $\mathcal{U}(\mathfrak{gl}_{n+1})$ .
- The map  $u \mapsto D_u$  is a linear map from  $\mathcal{U}_q(\mathfrak{gl}_n)$  to  $M$ .
- If  $u$  is central in  $\mathcal{U}_q(\mathfrak{gl}_n)$  then  $D_{uv} = D_u \circ D_v$  for any  $v \in \mathcal{U}_q(\mathfrak{gl}_n)$ . In particular,  $u \mapsto D_u$  is an algebra homomorphism from  $Z(\mathcal{U}_q(\mathfrak{gl}_n))$  to  $M$ .
- For any finite-dimensional representation  $W$ ,

$$D_u \text{Tr}_W(x^E) = \text{Tr}_W(ux^E). \quad (13)$$

- If  $u = 1$  is the identity element of  $\mathcal{U}_q(\mathfrak{gl}_n)$ , then  $D_1$  is the identity operator.

## 4 Quantum Random Walks on Quantum Groups

The first thing that needs to be defined is the states. Given  $D \in M$  and some  $\chi \in \mathbb{C}[[x_1, x_2, \dots]]$  define

$$\langle D \rangle_\chi = (D\chi)(1, 1, \dots)$$

In this paper,  $\chi$  will always be chosen so that the states are finite. Under the map  $u \mapsto D_u$ , this pulls back to a state on  $\mathcal{U}_q(\mathfrak{gl}_n)$  in the sense that

$$\langle u \rangle_\chi = (D_u \chi)(1, 1, \dots) \quad (14)$$

For  $t \geq 0$  let  $\chi_t(x_1, \dots, x_n)$  denote

$$\chi_t(x_1, \dots, x_n) = e^{t(x_1 - 1 + \dots + x_n - 1)} = e^{-tn} e^{t(x_1 + \dots + x_n)}$$

and to simplify notation let  $\langle \cdot \rangle_t = \langle \cdot \rangle_{\chi_t}$ . Given  $u_1 \in \mathcal{U}_{q_1}(\mathfrak{gl}_{n_1}), \dots, u_r \in \mathcal{U}_{q_r}(\mathfrak{gl}_{n_r})$ , use the formal notation

$$\langle u_1 \cdot u_2 \cdot \dots \cdot u_r \rangle_\chi = \langle D_{u_1} \circ \dots \circ D_{u_r} \rangle_\chi.$$

Note that because  $u_1, \dots, u_r$  are elements of different algebras, multiplication between them is not well-defined. However, the composition of the operators  $D_{u_1}, \dots, D_{u_r}$  is well-defined.

Now that the states have been defined, we define the non-commutative random walk. Fix times  $t_1 < t_2 < \dots$ . Let  $\mathcal{W}$  be the infinite tensor product  $M^{\otimes \infty}$  with respect to  $\langle \cdot \rangle_{t_1} \otimes \langle \cdot \rangle_{t_2 - t_1} \otimes \dots$ . For  $n \geq 1$  define the morphism  $j_{t_n} : \mathcal{U}_q(\mathfrak{gl}_n) \rightarrow \mathcal{W}$  to be the map

$$j_{t_n}(u) = D^{\otimes n}(\Delta^{(n-1)}u) \otimes \text{Id}^{\otimes \infty} = D_{(u_1)} \otimes \dots \otimes D_{(u_n)} \otimes \text{Id}^{\otimes \infty}.$$

and let  $\mathcal{W}_n$  be the subalgebra generated by the images of  $j_{t_1}, \dots, j_{t_n}$ . Let  $P_t$  be the non-commutative Markov operator on  $\mathcal{U}_q(\mathfrak{gl}_n)$  defined by  $P_t = (\text{id} \otimes \langle \cdot \rangle_t) \circ \Delta$ . We prove

**Theorem 4.1.** Assume that  $q$  is not a root of unity. Then

(1) The maps  $(j_n)$  are related to  $P_t$  by

$$\omega(j_n(X)w) = \omega(j_{n-1}(P_{t_n-t_{n-1}}X)w), \quad X \in \mathcal{U}_q(\mathfrak{gl}_n), \quad w \in \mathcal{W}_{n-1}.$$

(2) The non-commutative Markov operators preserve the states in the sense that

$$\langle P_t u \rangle_{\chi_\lambda} = \langle u_{(1)} \rangle_{\chi_\lambda} \langle u_{(2)} \rangle_t = \langle u \rangle_{\chi_\lambda \chi_t}$$

and satisfy the semi-group property  $P_{t+s} = P_t \circ P_s$ ,

(3) The pullback of  $\omega$  under  $j_n$  is the state  $\langle \cdot \rangle$  on  $\mathcal{U}_q(\mathfrak{gl}_n)$ , i.e.  $\langle X \rangle_{t_n} = \omega(j_n(X))$ .

(4) If  $X$  is central, then for  $n \leq m$  we have

$$\omega(j_n(X)j_m(Y)) = \langle X \cdot P_{t_m-t_n}Y \rangle_{t_m}$$

*Proof.* (1) The proof is similar to Theorem 4.1(1) from [19], which was itself based off of Proposition 3.1 from [9]. The left-hand-side is

$$\begin{aligned} \omega((\pi^{\otimes n-1} \otimes \pi)\Delta X w) &= \sum_{(X)} \omega(\pi^{\otimes n-1}(X_{(1)}) \otimes \pi(X_{(2)})w) \\ &= \sum_{(X)} \omega(\pi^{\otimes n-1}(X_{(1)})w) \langle X_{(2)} \rangle_{t_n-t_{n-1}} \end{aligned}$$

The right-hand-side is

$$\sum_{(X)} \omega(j_{n-1}(\langle X_{(2)} \rangle_{t_n-t_{n-1}} X_{(1)})w) = \sum_{(X)} \omega(j_{n-1}(X_{(1)})w) \langle X_{(2)} \rangle_{t_n-t_{n-1}}.$$

So the two sides are equal. Note that this argument did not assume that  $j_n$  is a homomorphism.

(2) First we show

$$\langle u_{(1)} \rangle_{\chi_\lambda} \langle u_{(2)} \rangle_t = \langle u \rangle_{\chi_\lambda \chi_t}$$

Because  $\{\chi_\lambda : \lambda \in \mathbb{GT}_n\}$  is a linear basis for the space of symmetric functions, by (13) it suffices to show that

$$\text{Tr}|_{V_\lambda} (u_{(1)} x^h) \text{Tr}|_{V_\mu} (u_{(2)} x^h) = \text{Tr}|_{V_\lambda \otimes V_\mu} (u x^h).$$

But this is true because the co-product is what defines the action on tensor powers of representations.

Now, by the co-associativity property,

$$(\text{id} \otimes \Delta) \circ \Delta(u) = (\Delta \otimes \text{id}) \circ \Delta(u) \implies u_{(1)} \otimes u_{(21)} \otimes u_{(22)} = u_{(11)} \otimes u_{(12)} \otimes u_{(2)}$$

Therefore

$$\begin{aligned} P_t \circ P_s(u) &= P_t(u_{(1)} \langle u_{(2)} \rangle_s) = u_{(11)} \langle u_{(12)} \rangle_t \langle u_{(2)} \rangle_s = (\text{id} \otimes \langle \cdot \rangle_t \otimes \langle \cdot \rangle_s)(u_{(11)} \otimes u_{(12)} \otimes u_{(2)}) \\ &= u_{(1)} \langle u_{(21)} \rangle_t \langle u_{(22)} \rangle_s = u_{(1)} \langle u_{(2)} \rangle_{t+s} = P_{t+s}(u) \end{aligned}$$

(3) By repeatedly applying (1) with  $w = 1$ , we have

$$\omega(j_n(X)) = \omega(j_{n-1}(P_{t_n-t_{n-1}}X)) = \dots = \omega(j_1(P_{t_2-t_1} \circ \dots \circ P_{t_n-t_{n-1}}(X)))$$

By the definition of  $\omega$  and  $j_1$ , and by applying (2) this equals

$$\langle P_{t_2-t_1} \circ \dots \circ P_{t_n-t_{n-1}}(X) \rangle_{t_1} = \langle P_{t_n-t_{n-1}}(X) \rangle_{t_1} = \langle X \rangle_{t_n}$$

(4) By repeated applications of (2), and then (1),

$$\omega(j_n(X)j_m(Y)) = \omega(j_n(X)j_n(P_{t_{n+1}-t_n} \circ \dots \circ P_{t_m-t_{m-1}}(Y))) = \omega(j_n(X)j_n(P_{t_m-t_n}(Y)))$$

Because  $X$  is central,  $D_{Xu} = D_x \circ D_u$  for any  $u$ , finishing the proof.  $\square$

**Remark.** There is a  $q$ -deformation of the algebra of functions which has a natural pairing with  $\mathcal{U}_q(\mathfrak{gl}_n)$ ; see e.g. Chapter 3 of [18]. One could instead define  $M$  as operators on this algebra, but this approach would not be sufficient for the purposes of this paper, as the results here require different values of  $q$ .

Also note that in the usual definition of a non-commutative random walk, the maps  $j_n$  are required to be algebra homomorphisms, and not merely linear. Here, the  $j_n$  are only algebra morphisms when restricted to the center of  $\mathcal{U}_q(\mathfrak{gl}_n)$ . Nevertheless, the results here are sufficient to show asymptotics in section 6.

**Remark.** Note that in the  $q = 1$  case, the non-commutative Markov operator  $P_t$  preserves the center (see Theorem 4.1(5) of [19] or Proposition 4.3 of [9]). We will see below that this is not true for general  $q$ . However, one could use the *quantum trace*

$$\langle u \rangle_{\chi_V}^{(q)} = \text{Tr}|_V (uq^{-2\rho})$$

where

$$2\rho = (n-1)E_{11} + (n-3)E_{22} + \cdots + (1-n)E_{nn}.$$

Then

$$\langle uv \rangle^{(q)} = \text{Tr} (uvq^{-2\rho}) = \text{Tr} (v \cdot q^{-2\rho}u) = \langle v \cdot q^{-2\rho}uq^{2\rho} \rangle^{(q)}.$$

By 4.9(1) of [15],  $S^2(u) = q^{-2\rho}uq^{2\rho}$ . Thus, if  $P_t^{(q)} = (\text{id} \otimes \langle \cdot \rangle_t^{(q)}) \circ \Delta$ , then Proposition 1.2(1) of [10] implies that  $P_t^{(q)}(u)$  is central if  $u$  is central. Note that when  $q = 1$ , then the quantum trace reduces to the usual trace.

## 5 Connections to random surface growth

In this section, we will show the relationship between the non-commutative random walk and a  $(2+1)$ -dimensional random surface growth model. First, here is a description of the model, which was introduced in [7].

### 5.1 Random surface growth

Consider the two-dimensional lattice  $\mathbb{Z} \times \mathbb{Z}_+$ . On each horizontal level  $\mathbb{Z} \times \{n\}$  there are exactly  $n$  particles, with at most one particle at each lattice site. Let  $\tilde{X}_1^{(n)} > \cdots > \tilde{X}_n^{(n)}$  denote the  $x$ -coordinates of the locations of the  $n$  particles. Additionally, the particles need to satisfy the *interlacing property*  $\tilde{X}_{i+1}^{(n+1)} < \tilde{X}_i^{(n)} \leq \tilde{X}_i^{(n+1)}$ . The particles can be viewed as a random stepped surface, see Figure 1. This can be made rigorous by defining the height function at  $(x, n)$  to be the number of particles to the right of  $(x, n)$ .

The dynamics on the particles are as follows. The initial condition is the *densely packed* initial condition,  $\tilde{X}_i^{(n)} = -i + 1, 1 \leq i \leq n$ . Each particle has a clock with exponential waiting time of rate 1, with all clocks independent of each other. When the clock rings, the particle attempts to jump one step to the right. However, it must maintain the interlacing property. This is done by having particles push particles above it, and jumps are blocked by particles below it. One can think of lower particles as being more massive. See Figure 2 for an example.

It turns out to be more convenient to use the co-ordinates  $X_i^{(n)} = \tilde{X}_i^{(n)} + i - 1$ . Then on each level,  $X_1^{(n)} \geq \cdots \geq X_n^{(n)}$  and the interlacing property becomes  $X_{i+1}^{(n+1)} \leq X_i^{(n)} \leq X_i^{(n+1)}$ . The initial condition is  $X_i^{(n)}(0) = 0$ .

Review some information about these probability measures and dynamics. By a result from [7, 8],

$$e^{t\text{Tr}(U - \text{Id})} = \sum_{\mu} \text{Prob} \left( X^{(N)}(t) = \mu \right) \frac{\chi_{\mu}(U)}{\dim \mu}$$

where  $\chi_\mu$  and  $\dim \mu$  are the character and dimension of the highest weight representation  $\mu$ . By Theorem 3.1 of [20] (with  $\theta = (1, \dots, 1)$  in the statement of that theorem), for  $t \geq s \geq 0$ ,

$$\mathbb{P}\left(X^{(N)}(t) = \tau | X^{(N)}(s) = \lambda\right) = \sum_{\mu} \mathbb{P}\left(X^{(N)}(t-s) = \mu\right) c_{\lambda\mu}^{\tau} \frac{\dim \tau}{\dim \lambda \dim \mu}.$$

where  $c_{\lambda\mu}^{\tau}$  are the Littlewood–Richardson coefficients defined by

$$\chi_{\lambda} \cdot \chi_{\mu} = \sum_{\tau} c_{\lambda\mu}^{\tau} \chi_{\tau}$$

And therefore

$$\begin{aligned} e^{t\text{Tr}(U-\text{Id})} \frac{\chi_{\lambda}}{\dim \lambda} &= \sum_{\mu} \text{Prob}\left(X^{(N)}(t) = \mu\right) \frac{\chi_{\mu} \cdot \chi_{\lambda}}{\dim \mu \dim \lambda} \\ &= \sum_{\mu, \tau} \text{Prob}\left(X^{(N)}(t) = \mu\right) \frac{\chi_{\tau}}{\dim \tau} c_{\lambda\mu}^{\tau} \frac{\dim \tau}{\dim \mu \dim \lambda} \\ &= \sum_{\tau} \frac{\chi_{\tau}}{\dim \tau} \mathbb{P}\left(X^{(N)}(s+t) = \tau | X^{(N)}(s) = \lambda\right) \end{aligned} \quad (15)$$

Furthermore, for all  $t \geq 0$ ,

$$\mathbb{P}\left(X^{(M)}(t) = \lambda^{(M)} | X^{(N)}(t) = \lambda^{(N)}\right) = \Lambda\left(\lambda^{(N)}, \lambda^{(M)}\right), \quad \forall M \leq N \quad (16)$$

where recall that  $\Lambda$  was defined in (12).

## 5.2 Restriction to center

Let  $Q_t$  be the Markov operator of the particle system, which defines an operator  $Q_t$  on  $\text{Fun}(\mathbb{GT}_n)$  by

$$Q_t f(\lambda) := \sum_{\mu \in \mathbb{GT}_n} Q_t(\lambda \rightarrow \mu) f(\mu).$$

Given  $u \in \mathcal{U}_q(\mathfrak{gl}_n)$ , there is a corresponding observable  $O_u$  on  $\mathbb{GT}_n$  given by

$$O_u(\lambda) := \frac{1}{\dim \lambda} \text{Tr}|_{V_{\lambda}} u \stackrel{(13)}{=} \frac{1}{\dim \lambda} D_u \chi_{\lambda}(1, \dots, 1) \stackrel{(14)}{=} \langle u \rangle_{\chi_{\lambda}/\dim \lambda}, \quad (17)$$

where  $\chi_{\lambda}$  was defined in (11). Observe that the map  $O : \mathcal{U}_q(\mathfrak{gl}_n) \rightarrow \text{Fun}(\mathbb{GT}_n)$  is a linear map. Let  $\mathcal{F} \subset \text{Fun}(\mathbb{GT}_n)$  be the image of  $O$ .

If  $m \leq n$  and  $u \in \mathcal{U}(\mathfrak{gl}_m)$ , then define  $O_u(\lambda)$  on  $\lambda \in \mathbb{GT}_n$  by

$$O_u(\lambda) = \sum_{\mu \in \mathbb{GT}_m} \Lambda(\lambda, \mu) O_u(\mu). \quad (18)$$

By the definition of  $\Lambda$ , (17) still holds.

**Proposition 5.1.** (1) For all  $u \in \mathcal{U}_q(\mathfrak{gl}_n)$  and  $t \geq 0$ ,

$$\langle u \rangle_t = \mathbb{E}[O_u(\lambda(t))]$$

(2) If  $f = O_u \in \mathcal{F}$ , then  $Q_t f = O_{P_t u}$ . In particular,  $Q_t$  preserves the image of  $O$ .

*Proof.* (1) By definition

$$\begin{aligned}
\langle u \rangle_t &= \sum_{\mu \in \mathbb{GT}_n} \mathbb{P}(\lambda(t) = \mu) \frac{(D_u \chi_\mu)(1, \dots, 1)}{\dim \mu} \\
&= \sum_{\mu \in \mathbb{GT}_n} \mathbb{P}(\lambda(t) = \mu) \frac{\text{Tr}|_{V_\mu} u}{\dim \mu} \\
&= \sum_{\mu \in \mathbb{GT}_n} \mathbb{P}(\lambda(t) = \mu) O_u(\mu) \\
&= \mathbb{E}[O_u(\lambda(t))]
\end{aligned}$$

(2) By linearity and (15)

$$\langle u \rangle_{\chi_t \frac{\chi_\lambda}{\dim \lambda}} = \sum_{\tau} \mathbb{P}\left(X^{(N)}(t) = \tau | X^{(N)}(s) = \lambda\right) \langle u \rangle_{\frac{\chi_\tau}{\dim \tau}}$$

or equivalently (by Theorem 4.1(2))

$$O_{P_t u}(\lambda) = \sum_{\tau} \mathbb{P}\left(X^{(N)}(t) = \tau | X^{(N)}(s) = \lambda\right) O_u(\tau) = (Q_t O_u)(\lambda).$$

□

The next theorem shows the multi-level relationship between the QRWQG and the random surface growth. This is similar to Theorem 4.5 of [19]. However, the proof there is no longer valid because the center is not preserved unless  $q = 1$ . The extra ingredient here is (15), which had not been used previously.

**Theorem 5.2.** *Assume that  $q$  is not a root of unity. Suppose that  $N_1 \geq \dots \geq N_r, t_1 \leq \dots \leq t_r$ . Let  $X_j \in Z(\mathcal{U}_{q_j}(gl_{N_j}))$  for  $1 \leq j \leq r-1$  and  $Y_r \in \mathcal{U}_{q_r}(gl_{N_r})$ . Then*

$$\begin{aligned}
\omega(j_{t_1}(X_1) \cdots j_{t_{r-1}}(X_{r-1}) j_{t_r}(Y_r)) \\
= \mathbb{E}\left[O_{X_1}\left(\lambda^{(N_1)}(t_1)\right) \cdots O_{X_{r-1}}\left(\lambda^{(N_{r-1})}(t_{r-1})\right) O_{Y_r}\left(\lambda^{(N_r)}(t_r)\right)\right]
\end{aligned}$$

*Proof.* First, note that by Theorem 4.1(4),

$$\omega(j_{t_1}(X_1) \cdots j_{t_{r-1}}(X_{r-1}) j_{t_r}(Y_r)) = \langle X_1 P_{t_2-t_1} (X_2 P_{t_3-t_2} (X_3 \cdots P_{t_r-t_{r-1}} Y_r)) \rangle_{t_1}$$

For the remainder of the proof, proceed by induction on  $r$ . When  $r = 1$  the result is Proposition 5.1(2).

Assume the statement for some  $r$ . Then setting  $Y_r = X_r \cdot P_{t_{r+1}-t_r} Y_{r+1}$ , the induction hypothesis implies

$$\begin{aligned}
\langle X_1 P_{t_2-t_1} (X_2 P_{t_3-t_2} (X_3 \cdots P_{t_r-t_{r-1}} (X_r P_{t_{r+1}-t_r} Y_{r+1}))) \rangle_{t_1} \\
= \mathbb{E}\left[O_{X_1}\left(\lambda^{(N_1)}(t_1)\right) \cdots O_{X_{r-1}}\left(\lambda^{(N_{r-1})}(t_{r-1})\right) O_{Y_r}\left(\lambda^{(N_r)}(t_r)\right)\right]
\end{aligned}$$

By the definition of an expectation, this equals (where the summation over each  $\mu^{(m)}$  is over  $\mathbb{GT}_m$ )

$$\sum_{\mu^{(N_1)}, \dots, \mu^{(N_r)}} \mathbb{P}\left(\lambda^{(N_1)}(t_1) = \mu^{(N_1)}, \dots, \lambda^{(N_r)}(t_r) = \mu^{(N_r)}\right) O_{X_1}\left(\mu^{(N_1)}\right) \cdots O_{X_{r-1}}\left(\mu^{(N_{r-1})}\right) O_{Y_r}\left(\mu^{(N_r)}\right) \quad (19)$$

By the definition of  $O$  in (17), and the assumption that  $X_r$  is central,

$$O_{Y_r}(\mu^{(N_r)}) = \frac{\text{Tr}|_{V_{\mu^{(N_r)}}}(X_r \cdot P_{t_{r+1}-t_r} Y_{r+1})}{\dim \mu^{(N_r)}} = O_{X_r}(\mu^{(N_r)}) \frac{\text{Tr}|_{V_{\mu^{(N_r)}}}(P_{t_{r+1}-t_r} Y_{r+1})}{\dim \mu^{(N_r)}}$$

Furthermore, setting  $\tilde{\chi}_\mu = \chi_\mu / \dim \mu$ ,

$$\begin{aligned} & \left( \dim \mu^{(N_r)} \right)^{-1} \text{Tr}|_{V_{\mu^{(N_r)}}}(P_{t_{r+1}-t_r} Y_{r+1}) = \langle P_{t_{r+1}-t_r} Y_{r+1} \rangle_{\tilde{\chi}_{\mu^{(N_r)}}} = \langle Y_{r+1} \rangle_{\tilde{\chi}_{\mu^{(N_r)}} \chi_{t_{r+1}-t_r}} \\ & \stackrel{(15)}{=} \sum_{\nu^{(N_r)}} \mathbb{P} \left( \lambda^{(N_r)}(t_{r+1}) = \nu^{(N_r)} | \lambda^{(N_r)}(t_r) = \mu^{(N_r)} \right) O_{Y_{r+1}} \left( \nu^{(N_r)} \right) \\ & \stackrel{(18)}{=} \sum_{\nu^{(N_r)}, \mu^{(N_{r+1})}} \mathbb{P} \left( \lambda^{(N_r)}(t_{r+1}) = \nu^{(N_r)} | \lambda^{(N_r)}(t_r) = \mu^{(N_r)} \right) O_{Y_{r+1}} \left( \mu^{(N_{r+1})} \right) \Lambda \left( \nu^{(N_r)}, \mu^{(N_{r+1})} \right) \end{aligned}$$

Therefore, by (16) the expression (19) equals

$$\begin{aligned} & \sum_{\substack{\mu^{(N_1)}, \dots, \mu^{(N_r)} \\ \nu^{(N_r)}, \mu^{(N_{r+1})}}} \mathbb{P} \left( \lambda^{(N_1)}(t_1) = \mu^{(N_1)}, \dots, \lambda^{(N_r)}(t_r) = \mu^{(N_r)}, \lambda^{(N_r)}(t_{r+1}) = \nu^{(N_r)}, \lambda^{(N_{r+1})}(t_{r+1}) = \mu^{(N_{r+1})} \right) \\ & \quad \times O_{X_1} \left( \mu^{(N_1)} \right) \cdots O_{X_{r-1}} \left( \mu^{(N_{r-1})} \right) O_{X_r} \left( \mu^{(N_r)} \right) O_{Y_{r+1}} \left( \mu^{(N_{r+1})} \right) \end{aligned}$$

Because there is no observable in  $\nu^{(N_r)}$ , the sum over  $\nu^{(N_r)}$  can be eliminated, completing the proof.  $\square$

Although there is not a rigorous way to multiply elements of  $\mathcal{U}_q(\mathfrak{gl}_n)$  and  $\mathcal{U}_{\tilde{q}}(\mathfrak{gl}_n)$ , it is not unreasonable to conjecture that the results in this section should still be true if the multiplication is interpreted formally. Here is a (numeric) example of how to do this.

**Example 1** Consider an irreducible representation of  $\mathcal{U}_q(\mathfrak{gl}_2)$  with highest weight  $(\lambda_1, \lambda_2)$ . The weights can be written as  $(\lambda_1 - j, \lambda_2 + j)$  for  $0 \leq j \leq \lambda_1 - \lambda_2$ . One can check that  $E_{21}E_{12} \in \mathcal{U}_q(\mathfrak{gl}_n)$  acting on  $(\lambda_1 - j, \lambda_2 + j)$  multiplies by the constant

$$[\lambda_1 - \lambda_2]_q + [\lambda_1 - \lambda_2 - 2]_q + \dots + [\lambda_1 - \lambda_2 - 2j + 2]_q.$$

So that  $\tilde{q}^{E_{11}+E_{22}} E_{21}E_{12}$  with the quantum trace  $\text{Tr}_s$  acts as the observable

$$\mathcal{O} = \frac{1}{\lambda_1 - \lambda_2 + 1} \sum_{j=1}^{\lambda_1 - \lambda_2} s^{\lambda_1 - \lambda_2 - 2j} \tilde{q}^{\lambda_1 + \lambda_2} ([\lambda_1 - \lambda_2]_q + [\lambda_1 - \lambda_2 - 2]_q + \dots + [\lambda_1 - \lambda_2 - 2j + 2]_q)$$

Now for  $\tilde{q} = -0.27 + 3i, q = 0.8, s = 0.6, t = 0.31, (a_1, a_2) = (3, 1)$ , the determinantal formula for  $\mathbb{E}[O(t)]$  from section 2.3 of [7] predicts

$$\begin{aligned} & e^{-2t} \sum_{\lambda_1=a_1}^{\infty} \sum_{\lambda_2=a_2}^{\lambda_1} \mathcal{O} \cdot (\lambda_1 - \lambda_2 + 1) \det \left( \begin{array}{cc} \frac{t^{\lambda_1}}{\lambda_1!} & \frac{t^{\lambda_1+1}}{(\lambda_1+1)!} \\ \frac{t^{\lambda_2-1}}{(\lambda_2-1)!} & \frac{t^{\lambda_2}}{\lambda_2!} \end{array} \right) \\ & \quad \approx -0.02788676811357415 - 0.002852163596477639i \end{aligned}$$

for the summation over  $\lambda_1$  up to  $50 \approx \infty$ .

Formally, the co-product applied to  $\tilde{q}^{E_{11}+E_{22}} E_{21}E_{12}$  yields

$$\Delta(\tilde{q}^{E_{11}+E_{22}} E_{21}E_{12}) = \tilde{q}^{E_{11}+E_{22}} E_{21}E_{12} \otimes \tilde{q}^{E_{11}+E_{22}} q^{E_{22}-E_{11}} + \tilde{q}^{E_{11}+E_{22}} q^{E_{11}-E_{22}} \otimes \tilde{q}^{E_{11}+E_{22}} E_{21}E_{12} \quad (20)$$

By applying  $\langle \cdot \rangle_t \otimes \langle \cdot \rangle_\epsilon$  to both sides and taking  $\epsilon \rightarrow 0$ , Theorem 4.1(2) implies that  $\langle \tilde{q}^{E_{11}+E_{22}} E_{21} E_{12} \rangle_t$  with the quantum trace at  $s$  solves the differential equation

$$y'(t) = (\tilde{q}q^{-1}s - 1 + \tilde{q}qs^{-1} - 1)y(t) + \tilde{q}s^{-1} \exp(t(\tilde{q}qs - 1 + \tilde{q}q^{-1}s^{-1} - 1)), \quad y(0) = 0$$

which is solved by

$$y(t) = e^{t(\tilde{q}qs^{-1} + \tilde{q}q^{-1}s - 2)} q(e^{\tilde{q}q^{-1}s^{-1}t(q^2-1)(s^2-1)} - 1)(q^2 - 1)^{-1}(s^2 - 1)^{-1}.$$

Furthermore, applying  $(\text{id} \otimes \langle \cdot \rangle_t^{(s)})$  to (20)

$$\begin{aligned} P_t^{(s)}(\tilde{q}^{E_{11}+E_{22}} E_{21} E_{12}) &= \langle \tilde{q}^{E_{11}+E_{22}} q^{E_{22}-E_{11}} \rangle_t^{(s)} \tilde{q}^{E_{11}+E_{22}} E_{21} E_{12} + \langle \tilde{q}^{E_{11}+E_{22}} E_{21} E_{12} \rangle_t^{(s)} \tilde{q}^{E_{11}+E_{22}} q^{E_{11}-E_{22}} \\ &= e^{t(\tilde{q}q^{-1}s + \tilde{q}qs^{-1} - 2)} \tilde{q}^{E_{11}+E_{22}} E_{21} E_{12} + y(t) \tilde{q}^{E_{11}+E_{22}} q^{E_{11}-E_{22}} \end{aligned}$$

which predicts

$$\begin{aligned} e^{t(\tilde{q}q^{-1}s + \tilde{q}qs^{-1} - 2)} \cdot \mathcal{O} + \frac{1}{\lambda_1 - \lambda_2 + 1} \sum_{j=0}^{\lambda_1 - \lambda_2} s^{\lambda_1 - \lambda_2 - 2j} \tilde{q}^{\lambda_1 + \lambda_2} q^{\lambda_1 - j - (\lambda_2 + j)} \\ = -0.02788676811357414 - 0.002852163596477645i \end{aligned}$$

which matches to 17 decimal points.

## 6 Asymptotic Gaussian Fluctuations

By (46) in [13], the element

$$C^{(n)} = C_q^{(n)} := \sum_{i=1}^n q^{2i-2n} q^{2E_{ii}} + (q - q^{-1})^2 \sum_{1 \leq i < j \leq n} q^{2j-2n-1} q^{E_{ii}+E_{jj}} E_{ij} E_{ji}$$

is central in  $\mathcal{U}_q(\mathfrak{gl}_n)$ . When acting on the lowest weight vector of  $V_\lambda$ , the second term vanishes, so  $C_q^{(n)}$  acts as the constant (see also (51) in [13])

$$\sum_{i=1}^n q^{2(\lambda_i - i + n)} = \sum_{i=1}^n \sum_{k=0}^{\infty} \frac{(2h)^k (\lambda_i - i + n)^k}{k!} =: \sum_{k=0}^{\infty} \frac{(2h)^k}{k!} \Psi_k^{(n)}(\lambda) \quad (21)$$

where  $q = \exp h$ . By previously known results ([6, 7, 19]), there are fixed-time asymptotics: if  $N_j = [\eta_j L], t = [\tau L]$  then  $\mathbb{E} \Psi_k^{(N)}(\lambda(t)) \sim L^{k+1}$  (where  $\lambda(t)$  is distributed as  $X^{(N)}(t)$ ) and

$$\left( \frac{\Psi_{k_1}^{(N_1)}(\lambda(t)) - \mathbb{E} \Psi_{k_1}^{(N_1)}(\lambda(t))}{L^{k_1}}, \dots, \frac{\Psi_{k_r}^{(N_r)}(\lambda(t)) - \mathbb{E} \Psi_{k_r}^{(N_r)}(\lambda(t))}{L^{k_r}} \right) \rightarrow (\xi_1, \dots, \xi_r) \quad (22)$$

where  $(\xi_1, \dots, \xi_r)$  is a Gaussian vector with mean zero covariance

$$\mathbb{E}[\xi_i \xi_j] = \mathbb{E}[\mathfrak{G}(k_i, \eta_i, \tau) \mathfrak{G}(k_j, \eta_j, \tau)].$$

By (21), this suggests that  $q_j$  should depend on  $L$  as  $q_j = \exp h_j/L$ . This scaling also suggests that  $\langle C_q^{(N)} \rangle_t$  should be of order  $\sim L$  with fluctuations of constant order, which will be confirmed below.

For multi-time asymptotics, it is also necessary to find the states of each monomial in  $C_q^{(n)}$ . Below, recall that

$${}_1F_1(-m; 2; -x) = \sum_{r=1}^{m+1} \binom{m}{r-1} \frac{x^{r-1}}{r!}.$$

**Proposition 6.1.** Assume that  $q$  is not a root of unity. For  $i < j$ ,

$$\langle q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \rangle_\gamma = q\gamma e^{\gamma(q^2-1)} {}_1F_1(-(j-i-1); 2; -(q-q^{-1})^2\gamma) =: f_{j-i}(\gamma).$$

*Proof.* By (10), for  $1 \leq i < j \leq n$

$$\begin{aligned} \Delta(q^{E_{ii}+E_{jj}} E_{ij} E_{ji}) &= q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \otimes q^{2E_{jj}} + q^{2E_{ii}} \otimes q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \\ &\quad + (q-q^{-1})^2 \sum_{r=i+1}^{j-1} \sum_{k=i+1}^{j-1} q^{E_{ii}+E_{rr}} E_{ir} E_{ki} \otimes q^{E_{ii}+E_{jj}} E_{rj} q^{E_{kk}-E_{ii}} E_{jk} \end{aligned}$$

Hence, by Theorem 4.1(2),

$$\begin{aligned} \langle q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \rangle_{\gamma+\epsilon} &= \langle q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \rangle_\gamma \langle q^{2E_{ii}} \rangle_\epsilon + \langle q^{2E_{jj}} \rangle_\gamma \langle q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \rangle_\epsilon \\ &\quad + (q-q^{-1})^2 q^{-1} \sum_{r=i+1}^{j-1} \langle q^{E_{rr}+E_{jj}} E_{rj} E_{jr} \rangle_\gamma \langle q^{E_{ii}+E_{rr}} E_{ir} E_{ri} \rangle_\epsilon \end{aligned} \quad (23)$$

where we have used  $q^{E_{rr}-E_{ii}} E_{rj} = E_{rj} q^{E_{rr}-E_{ii}} q^{(\epsilon_r, \epsilon_r - \epsilon_{r+1})} = E_{rj} q^{E_{rr}-E_{ii}} q$  for  $r < j$ . In particular,

$$\begin{aligned} \langle q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \rangle_{\gamma+\epsilon} &= (1+\epsilon(q^2-1)) \langle q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \rangle_\gamma \\ &\quad + q\epsilon \langle q^{2E_{jj}} \rangle_\gamma + (q-q^{-1})^2 q^{-1} \sum_{r=i+1}^{j-1} \langle q^{E_{rr}+E_{jj}} E_{rj} E_{jr} \rangle_\gamma q\epsilon + O(\epsilon^2). \end{aligned}$$

Therefore,  $f_{j-i}(\gamma) := \langle q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \rangle_\gamma$  satisfies the differential equation

$$f'_{j-i}(\gamma) = (q^2-1)f_{j-i}(\gamma) + qe^{\gamma(q^2-1)} + (q-q^{-1})^2 \sum_{r=i+1}^{j-1} f_{j-r}(\gamma).$$

In general, if a family of functions  $\{f_m(t)\}$  satisfies the differential equation

$$f'_m(x) = af_m(x) + g(x) + b \sum_{i=1}^{m-1} f_i(x)$$

then using integrating factors shows that  $f_m$  is solved by

$$\begin{aligned} f_m(x) &= e^{ax} \int_0^x e^{-ax_1} \left( g(x_1) + b \sum_{i_1=1}^{m-1} f_{i_1}(x_1) \right) dx_1 \\ &= e^{ax} \int_0^x e^{-ax_1} g(x_1) dx_1 + e^{ax} b \sum_{i_1=1}^{m-1} \int_0^x \int_0^{x_1} e^{-ax_2} \left( g(x_2) + \sum_{i_2=1}^{i_1-1} f_{i_2}(x_2) \right) dx_2 dx_1 \\ &= e^{ax} \int_0^x e^{-ax_1} g(x_1) dx_1 + e^{ax} b \binom{m-1}{1} \int_0^x \int_0^{x_1} e^{-ax_2} g(x_2) dx_2 dx_1 + \dots \\ &= \sum_{r=1}^m e^{ax} \binom{m-1}{r-1} b^{r-1} \int_0^x \dots \int_0^{x_{r-1}} e^{-ax_r} g(x_r) dx_r \dots dx_1 \end{aligned}$$

Applying this with  $a = (q^2-1)$ ,  $b = (q-q^{-1})^2$  and  $g(t) = qe^{t(q^2-1)}$  yields

$$\begin{aligned} f_{ij}(\gamma) &= qe^{\gamma(q^2-1)} \sum_{r=1}^{j-i} \binom{j-i-1}{r-1} (q-q^{-1})^{2(r-1)} \frac{\gamma^r}{r!} \\ &= q\gamma e^{\gamma(q^2-1)} {}_1F_1(-(j-i-1); 2; -(q-q^{-1})^2\gamma) \end{aligned}$$

□



The following result is generalization for  $n = 2$  in [4] (see also chapter 13 in [2]).

**Proposition 6.2.** *Assume that  $q$  is not a root of unity. For any  $t \geq 0$ ,*

$$P_t(C^{(n)}) = e^{t(q^2-1)} \cdot C^{(n)} + \sum_{k=1}^{n-1} A_k(t) C^{(n-k)} \quad (24)$$

where

$$A_k(t) = q^{-1}(q - q^{-1})^2 f_k(t).$$

*Proof.* Since  $\Delta(q^{2E_{ii}}) = q^{2E_{ii}} \otimes q^{2E_{ii}}$ ,

$$P_t(q^{2E_{ii}}) = e^{t(q^2-1)} q^{2E_{ii}}$$

Now using (10), we have

$$\begin{aligned} P_t(q^{E_{ii}+E_{jj}} E_{ij} E_{ji}) &= e^{t(q^2-1)} q^{E_{ii}+E_{jj}} E_{ij} E_{ji} + \langle q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \rangle_t q^{2E_{ii}} \\ &\quad + q^{-1}(q - q^{-1})^2 \sum_{r=i+1}^{j-1} \langle q^{E_{rr}+E_{jj}} E_{rj} E_{jr} \rangle_t q^{E_{ii}+E_{rr}} E_{ir} E_{ri} \end{aligned}$$

where we have used  $q^{E_{rr}-E_{ii}} E_{rj} = E_{rj} q^{E_{rr}-E_{ii}} q^{(\epsilon_r, \epsilon_r - \epsilon_{r+1})} = E_{rj} q^{E_{rr}-E_{ii}} q$  for  $r < j$ . Because the term  $e^{t(q^2-1)}$  occurs as a coefficient in both  $q^{2E_{ii}}$  and  $q^{E_{ii}+E_{jj}} E_{ij} E_{ji}$ , we can write

$$\begin{aligned} P_t(C^{(n)}) &- e^{t(q^2-1)} \cdot C^{(n)} \\ &= (q - q^{-1})^2 \sum_{1 \leq i < j \leq n} q^{2j-2n-1} \left( f_{j-i}(t) q^{2E_{ii}} + q^{-1}(q - q^{-1})^2 \sum_{r=i+1}^{j-1} f_{j-r}(t) q^{E_{ii}+E_{rr}} E_{ir} E_{ri} \right) \\ &= (q - q^{-1})^2 \sum_{i=1}^{n-1} \left( \sum_{j=i+1}^n q^{2j-2n-1} f_{j-i}(t) \right) q^{2E_{ii}} \\ &\quad + (q - q^{-1})^4 \sum_{1 \leq i < r \leq n-1} \left( \sum_{j=r+1}^n q^{2j-2n-2} f_{j-r}(t) \right) q^{E_{ii}+E_{rr}} E_{ir} E_{ri}. \end{aligned}$$

Re-arrange the summation to note that

$$\begin{aligned} &\sum_{k=1}^{n-1} A_k(t) C^{(n-k)} \\ &= \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} A_k(t) q^{2i-2n+2k} q^{2E_{ii}} + (q - q^{-1})^2 \sum_{k=1}^{n-1} \sum_{1 \leq i < j \leq n-k} A_k(t) q^{2j-2n+2k-1} q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} A_k(t) q^{2i-2n+2k} q^{2E_{ii}} + (q - q^{-1})^2 \sum_{1 \leq i < j \leq n-1} \sum_{k=1}^{n-j} A_k(t) q^{2j-2n+2k-1} q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} A_k(t) q^{2i-2n+2k} q^{2E_{ii}} + (q - q^{-1})^2 \sum_{1 \leq i < r \leq n-1} \sum_{k=1}^{n-r} A_k(t) q^{2r-2n+2k-1} q^{E_{ii}+E_{rr}} E_{ir} E_{ri} \end{aligned}$$

And now setting  $k = j - i$  in the first sum and  $k = j - r$  in the second sum shows the result.  $\square$

Notice that in the scalings at the beginning of this section,  $\langle q^{E_{ii}+E_{jj}} E_{ij} E_{ji} \rangle_t$  is of order  $L$  and  $\langle q^{2E_{ii}} \rangle_t$  is of constant order. This implies that  $\langle C_q^{(N)} \rangle_t$  is of order  $L$ , as expected.

We can now state the convergence.

**Theorem 6.3.** Suppose that  $N_j = [\eta_j L]$ ,  $t_j = \tau_j L$  and  $q_j = \exp(h_j/L)$  for  $1 \leq j \leq r$ . Then as  $L \rightarrow \infty$ ,

$$\left( j_{t_1} \left( C_{q_1}^{(N_1)} - \left\langle C_{q_1}^{(N_1)} \right\rangle_{t_1} \right), \dots, j_{t_r} \left( C_{q_r}^{(N_r)} - \left\langle C_{q_r}^{(N_r)} \right\rangle_{t_r} \right) \right) \rightarrow (\mathcal{G}(h_1, \eta_1, \tau_1), \dots, \mathcal{G}(h_r, \eta_r, \tau_r))$$

with respect to the state  $\omega(\cdot)$ .

*Proof.* Because  $P_t C^{(n)}$  is a linear combination of  $C^{(k)}$  for  $1 \leq k \leq n$ , repeated applications of Theorem 4.1(4) shows that the multi-time fluctuations can be written as a linear combination of fixed-time fluctuations of central elements. Each central element has a series of the form (21), so it follows from (22) that the convergence will be to some Gaussian vector. It remains to show that the covariance is that of  $\mathcal{G}$ .

The theorem for fixed-time follows from Proposition 2.1 and the discussion at the beginning of this section. By Theorem 4.1(4), it suffices to calculate the limit of

$$\left\langle \left( C_{q_1}^{(N_1)} - \left\langle C_{q_1}^{(N_1)} \right\rangle_{t_1} \right) \cdot \left( P_{t_2-t_1} C_{q_2}^{(N_2)} - \left\langle P_{t_2-t_1} C_{q_2}^{(N_2)} \right\rangle_{t_1} \right) \right\rangle_{t_1}.$$

By Proposition 6.2, this equals

$$\begin{aligned} e^{(t_2-t_1)(q_2^2-1)} & \left\langle \left( C_{q_1}^{(N_1)} - \left\langle C_{q_1}^{(N_1)} \right\rangle_{t_1} \right) \cdot \left( C_{q_2}^{(N_2)} - \left\langle C_{q_2}^{(N_2)} \right\rangle_{t_1} \right) \right\rangle_{t_1} \\ & + \sum_{k=1}^{N_2-1} A_k(t_2-t_1) \left\langle \left( C_{q_1}^{(N_1)} - \left\langle C_{q_1}^{(N_1)} \right\rangle_{t_1} \right) \cdot \left( C_{q_2}^{(N_2-k)} - \left\langle C_{q_2}^{(N_2-k)} \right\rangle_{t_1} \right) \right\rangle_{t_1}. \end{aligned}$$

If  $k$  depends on  $L$  as  $k = [\kappa L]$ , then

$$\begin{aligned} A_{n-k}^{(n)}(t) &= q^{-1}(q - q^{-1})^2 \cdot f_k(t) \approx (2h/L)^2 e^{2h\tau} \sum_{r=1}^{\kappa L} \frac{(\kappa L - 1)^{r-1}}{(r-1)!} (2h/L)^{2(r-1)} \frac{\tau^r L^r}{r!} \\ &\approx L^{-1} e^{2h\tau} \sum_{r=1}^{\infty} \frac{\kappa^{r-1}}{(r-1)!} (2h)^{2r} \frac{\tau^r}{r!} \\ &= L^{-1} e^{2h\tau} \cdot \frac{2h\sqrt{\tau}}{\sqrt{\kappa}} I_1(4h\sqrt{\kappa\tau}) \end{aligned}$$

where  $I_n(x)$  is the modified Bessel function of the first kind:

$$I_n(x) = \sum_{r=1}^{\infty} \frac{1}{(r-1)!(r-1+n)!} \left( \frac{x}{2} \right)^{2(r-1)+n} = \frac{1}{2\pi i} \oint e^{(x/2)(t+t^{-1})} t^{-n-1} dt,$$

where the contour encloses the origin in a counterclockwise direction. The sum over  $k$  becomes a Riemann sum for an integral over  $\kappa$ , so therefore the asymptotic limit is

$$\begin{aligned} e^{2(\tau_2-\tau_1)h_2} \mathbb{E}[\mathcal{G}(h_1, \eta_1, \tau_1) \mathcal{G}(h_2, \eta_2, \tau_1)] \\ + e^{2(\tau_2-\tau_1)h_2} \cdot 2h_2 \sqrt{\tau_2 - \tau_1} \int_0^{\eta_2} \kappa^{-1/2} \cdot \frac{1}{2\pi i} \oint e^{2h_2 \sqrt{\kappa} \sqrt{\tau_2 - \tau_1} (t+t^{-1})} t^{-2} dt. \end{aligned}$$

By (3), this equals  $\mathbb{E}[\mathcal{G}(h_1, \eta_1, \tau_1) \mathcal{G}(h_2, \eta_2, \tau_2)]$ , which completes the proof.  $\square$

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Figure 1: The particles as a stepped surface. The lattice is shifted to make the visualization easier.

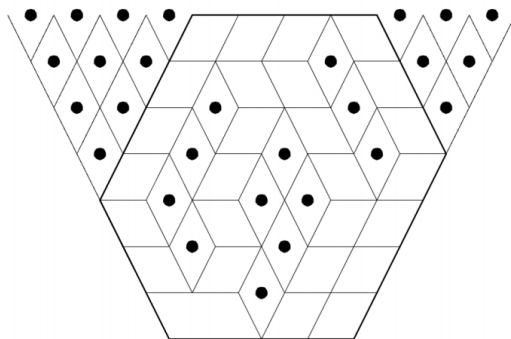


Figure 2: The red particle makes a jump. If any of the black particles attempt to jump, their jump is blocked by the particle below and to the right, and nothing happens. White particles are not blocked.

